Calculus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 8

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Textbook Sections: 7.1

Today: We begin the next "main section" of our course, which consists of a toolbox of useful techniques for computing and, more generally, analyzing the behavior of integrals.

We'll kick this off with what is certainly one of the most useful, and most important, techniques:

Integration by Parts (IBP)

"Motivation:" Recall that u-substitution is inspired by "doing the chain rule for derivatives in reverse".

Let's try cooking up "the reverse of the product rule", and see if it helps us with doing any new integrals.

Recall: Let $f(x)$ and $g(x)$ be differentiable functions on $[a,b]$. Then

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

Immediately undoing this by integrating, we get

indefinite version:
$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

definite version:
$$f(x)g(x)\Big|_a^b = \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx$$

Notes:

1. More commonly, people write this in the order

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

and likewise for the definite version.

2. Setting $u = f(x)$ and $v = g(x)$, we have

$$du = f'(x)\,dx, \quad dv = g'(x)\,dx$$

so that the formula looks like

$$\int uv' = uv - \int u'v.$$

In Practice: Let's suppose we encounter an integral in the following form:

$$\int \varphi(x)\xi(x)\,dx.$$

[Here, we are just assuming that the integrand is some product of functions φ and ξ.]

1. Try to identify if one of φ, ξ is easy to integrate, or if one becomes "simpler" when it is differentiated.

   ex: Lets suppose we are lucky, and recognize that φ has anti-derivative Φ.

2. Apply IBP: "Choose $u = \xi$ and $v = \Phi$"

$$\int \varphi(x)\xi(x)\,dx = \Phi(x)\xi(x) - \int \Phi(x)\xi'(x)\,dx$$

3. Now study the integral $\int \Phi(x)\xi'(x)\,dx$!

The main point: In the right circumstances, $\xi'(x)$ will be more simple than $\xi(x)$, and in lucky circumstances, this last integral will be easier than the original one!

⚠ It's easiest to get the hang of this technique by doing lots of practice problems.

A key lesson to keep in mind: getting good at IBP, like with most integration techniques, involves a lot of trial and error. Don't be afraid to jump into a calculation and try ideas out — if one approach doesn't work, try changing some of your choices around and find what works! Don't be afraid to use other techniques (like u-sub) at the same time!

Examples:

① $\int xe^x\,dx = xe^x - \int e^x = (x-1)e^x + C.$

② $\int \ln x\,dx = \int 1\cdot \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}dx = x\ln x - x + C$

③ $\int_0^3 t^2e^t\,dt = e^t t^2\Big|_0^3 - 2\int_0^3 te^t\,dt = 9e^3 - 2\left[te^t\Big|_0^3 - \int_0^3 e^t\,dt\right] = 9e^3 - 2\left[3e^3 - e^t\Big|_0^3\right] = 9e^3 - 6e^3 + 2e^3 - 2 = 5e^3 - 2$

④ $\int e^\theta\cos\theta\,d\theta = e^\theta\cos\theta + \int e^\theta\sin\theta = e^\theta\cos\theta + e^\theta\sin\theta - \int e^\theta\cos\theta$

⑤ $\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} y^3\cos(y^2)\,dy = \int_{\pi/2}^{\pi} \frac{1}{2}u\cos u\,du = \frac{1}{2}(u\sin u)\Big|_{\frac{\pi}{2}}^{\pi} - \frac{1}{2}\int_{\pi/2}^{\pi}\sin u\,du$

$= -\left(\frac{\pi}{4} + \frac{1}{2}\right)$

$u = y^2$

$du = 2y\,dy$

#74 on HW